

## SOME RELATIONS INVOLVING PARTIAL DERIVATIVES

### A-1 PARTIAL DERIVATIVES

In thermodynamics we are interested in continuous functions of three (or more) variables

$$\psi = \psi(x, y, z) \quad (\text{A.1})$$

If two independent variables, say  $y$  and  $z$ , are held constant,  $\psi$  becomes a function of only one independent variable  $x$ , and the derivative of  $\psi$  with respect to  $x$  may be defined and computed in the standard fashion. The derivative so obtained is called the *partial derivative* of  $\psi$  with respect to  $x$  and is denoted by the symbol  $(\partial\psi/\partial x)_{y,z}$  or simply by  $\partial\psi/\partial x$ . The derivative depends upon  $x$  and upon the values at which  $y$  and  $z$  are held during the differentiation; that is  $\partial\psi/\partial x$  is a function of  $x$ ,  $y$ , and  $z$ . The derivatives  $\partial\psi/\partial y$  and  $\partial\psi/\partial z$  are defined in an identical manner.

The function  $\partial\psi/\partial x$ , if continuous, may itself be differentiated to yield three derivatives which are called the *second partial derivatives* of  $\psi$

$$\begin{aligned} \frac{\partial}{\partial x} \left( \frac{\partial\psi}{\partial x} \right) &\equiv \frac{\partial^2\psi}{\partial x^2} \\ \frac{\partial}{\partial y} \left( \frac{\partial\psi}{\partial x} \right) &\equiv \frac{\partial^2\psi}{\partial y \partial x} \\ \frac{\partial}{\partial z} \left( \frac{\partial\psi}{\partial x} \right) &\equiv \frac{\partial^2\psi}{\partial z \partial x} \end{aligned} \quad (\text{A.2})$$

By partial differentiation of the functions  $\partial\psi/\partial y$  and  $\partial\psi/\partial z$ , we obtain other second partial derivatives of  $\psi$

$$\frac{\partial^2\psi}{\partial x \partial y} \quad \frac{\partial^2\psi}{\partial y^2} \quad \frac{\partial^2\psi}{\partial z \partial y} \quad \frac{\partial^2\psi}{\partial x \partial z} \quad \frac{\partial^2\psi}{\partial y \partial z} \quad \frac{\partial^2\psi}{\partial z^2}$$

It may be shown that under the continuity conditions that we have assumed for  $\psi$  and its partial derivatives the order of differentiation is immaterial, so that

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}, \quad \frac{\partial^2 \psi}{\partial x \partial z} = \frac{\partial^2 \psi}{\partial z \partial x}, \quad \frac{\partial^2 \psi}{\partial y \partial z} = \frac{\partial^2 \psi}{\partial z \partial y} \quad (\text{A.3})$$

There are therefore just six nonequivalent second partial derivatives of a function of three independent variables (three for a function of two variables, and  $\frac{1}{2}n(n+1)$  for a function of  $n$  variables).

## A-2 TAYLOR'S EXPANSION

The relationship between  $\psi(x, y, z)$  and  $\psi(x + dx, y + dy, z + dz)$ , where  $dx$ ,  $dy$ , and  $dz$  denote arbitrary increments in  $x$ ,  $y$ , and  $z$ , is given by Taylor's expansion

$$\begin{aligned} & \psi(x + dx, y + dy, z + dz) \\ &= \psi(x, y, z) + \left( \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial z} dz \right) + \frac{1}{2} \left[ \frac{\partial^2 \psi}{\partial x^2} (dx)^2 + \frac{\partial^2 \psi}{\partial y^2} (dy)^2 \right. \\ & \quad \left. + \frac{\partial^2 \psi}{\partial z^2} (dz)^2 + 2 \frac{\partial^2 \psi}{\partial x \partial y} dx dy + 2 \frac{\partial^2 \psi}{\partial x \partial z} dx dz + 2 \frac{\partial^2 \psi}{\partial y \partial z} dy dz \right] + \dots \end{aligned} \quad (\text{A.4})$$

This expansion can be written in a convenient symbolic form

$$\psi(x + dx, y + dy, z + dz) = e^{(dx(\partial/\partial x) + dy(\partial/\partial y) + dz(\partial/\partial z))} \psi(x, y, z) \quad (\text{A.5})$$

Expansion of the symbolic exponential according to the usual series

$$e^x = 1 + x + \frac{1}{2!}x^2 + \dots + \frac{1}{n!}x^n + \dots \quad (\text{A.6})$$

then reproduces the Taylor expansion (equation A.4)

### A-3 DIFFERENTIALS

The Taylor expansion (equation A.4) can also be written in the form

$$\begin{aligned} & \psi(x + dx, y + dy, z + dz) - \psi(x, y, z) \\ & = d\psi + \frac{1}{2!}d^2\psi + \dots + \frac{1}{n!}d^n\psi \dots \end{aligned} \quad (\text{A.7})$$

where

$$d\psi \equiv \frac{\partial\psi}{\partial x} dx + \frac{\partial\psi}{\partial y} dy + \frac{\partial\psi}{\partial z} dz \quad (\text{A.8})$$

$$\begin{aligned} d^2\psi &= \frac{\partial^2\psi}{\partial x^2}(dx)^2 + \frac{\partial^2\psi}{\partial y^2}(dy)^2 + \frac{\partial^2\psi}{\partial z^2}(dz)^2 + 2\frac{\partial^2\psi}{\partial x\partial y} dx dy \\ &+ 2\frac{\partial^2\psi}{\partial x\partial z} dx dz + 2\frac{\partial^2\psi}{\partial y\partial z} dy dz \end{aligned} \quad (\text{A.9})$$

and generally

$$d^n\psi = \left( dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right)^n \psi(x, y, z) \quad (\text{A.10})$$

These quantities  $d\psi, d^2\psi, \dots, d^n\psi, \dots$  are called the *first-, second-, and  $n$ th-order differentials* of  $\psi$ .

### A-4 COMPOSITE FUNCTIONS

Returning to the first-order differential

$$d\psi = \left( \frac{\partial\psi}{\partial x} \right)_{y,z} dx + \left( \frac{\partial\psi}{\partial y} \right)_{x,z} dy + \left( \frac{\partial\psi}{\partial z} \right)_{x,y} dz \quad (\text{A.11})$$

an interesting case arises when  $x, y,$  and  $z$  are not varied independently but are themselves considered to be functions of some variable  $u$ . Then

$$dx = \frac{dx}{du} du \quad dy = \frac{dy}{du} du \quad \text{and} \quad dz = \frac{dz}{du} du$$

whence

$$d\psi = \left[ \left( \frac{\partial\psi}{\partial x} \right)_{y,z} \frac{dx}{du} + \left( \frac{\partial\psi}{\partial y} \right)_{x,z} \frac{dy}{du} + \left( \frac{\partial\psi}{\partial z} \right)_{x,y} \frac{dz}{du} \right] du \quad (\text{A.12})$$

or

$$\frac{d\psi}{du} = \left(\frac{\partial\psi}{\partial x}\right)_{y,z} \frac{dx}{du} + \left(\frac{\partial\psi}{\partial y}\right)_{x,z} \frac{dy}{du} + \left(\frac{\partial\psi}{\partial z}\right)_{x,y} \frac{dz}{du} \quad (\text{A.13})$$

If  $x$  and  $y$  are functions of two (or more) variables, say  $u$  and  $v$ , then

$$dx = \left(\frac{\partial x}{\partial u}\right)_v du + \left(\frac{\partial x}{\partial v}\right)_u dv, \quad \text{etc.}$$

and

$$\begin{aligned} d\psi = & \left[ \left(\frac{\partial\psi}{\partial x}\right)_{y,z} \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial\psi}{\partial y}\right)_{x,z} \left(\frac{\partial y}{\partial u}\right)_v + \left(\frac{\partial\psi}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial u}\right)_v \right] du \\ & + \left[ \left(\frac{\partial\psi}{\partial x}\right)_{y,z} \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial\psi}{\partial y}\right)_{x,z} \left(\frac{\partial y}{\partial v}\right)_u + \left(\frac{\partial\psi}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial v}\right)_u \right] dv \end{aligned} \quad (\text{A.14})$$

or

$$d\psi = \left(\frac{\partial\psi}{\partial u}\right)_v du + \left(\frac{\partial\psi}{\partial v}\right)_u dv \quad (\text{A.15})$$

where

$$\left(\frac{\partial\psi}{\partial u}\right)_v = \left(\frac{\partial\psi}{\partial x}\right)_{y,z} \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial\psi}{\partial y}\right)_{x,z} \left(\frac{\partial y}{\partial u}\right)_v + \left(\frac{\partial\psi}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial u}\right)_v \quad (\text{A.16})$$

and similarly for  $(\partial\psi/\partial v)_u$ .It may happen that  $u$  is identical to  $x$  itself. Then

$$\left(\frac{\partial\psi}{\partial x}\right)_v = \left(\frac{\partial\psi}{\partial x}\right)_{y,z} + \left(\frac{\partial\psi}{\partial y}\right)_{x,z} \left(\frac{\partial y}{\partial x}\right)_v + \left(\frac{\partial\psi}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial x}\right)_v \quad (\text{A.17})$$

Other special cases can be treated similarly.

## A-5 IMPLICIT FUNCTIONS

If  $\psi$  is held constant, the variations of  $x$ ,  $y$ , and  $z$  are not independent, and the relation

$$\psi(x, y, z) = \text{constant} \quad (\text{A.18})$$

gives an implicit functional relation among  $x$ ,  $y$ , and  $z$ . This relation may be solved for one variable, say  $z$ , in terms of the other two

$$z = z(x, y) \quad (\text{A.19})$$

This function can then be treated by the techniques previously described to derive certain relations among the partial derivatives. However, a more direct method of obtaining the appropriate relations among the partial derivatives is merely to put  $d\psi = 0$  in equation A.8.

$$0 = \left(\frac{\partial\psi}{\partial x}\right)_{y,z} dx + \left(\frac{\partial\psi}{\partial y}\right)_{x,z} dy + \left(\frac{\partial\psi}{\partial z}\right)_{x,y} dz \quad (\text{A.20})$$

If we now put  $dz = 0$  and divide through by  $dx$ , we find

$$0 = \left(\frac{\partial\psi}{\partial x}\right)_{y,z} + \left(\frac{\partial\psi}{\partial y}\right)_{x,z} \left(\frac{\partial y}{\partial x}\right)_{\psi,z} \quad (\text{A.21})$$

in which the symbol  $(\partial y/\partial x)_{\psi,z}$  appropriately indicates that the implied functional relation between  $y$  and  $x$  is that determined by the constancy of  $\psi$  and  $z$ . Equation A.21 can be written in the convenient form

$$\left(\frac{\partial y}{\partial x}\right)_{\psi,z} = \frac{-(\partial\psi/\partial x)_{y,z}}{(\partial\psi/\partial y)_{x,z}} \quad (\text{A.22})$$

This equation plays a very prominent role in thermodynamic calculations. By successively putting  $dy = 0$  and  $dx = 0$  in equation A.20, we find the two similar relations

$$\left(\frac{\partial z}{\partial x}\right)_{\psi,y} = \frac{-(\partial\psi/\partial x)_{y,z}}{(\partial\psi/\partial z)_{x,y}} \quad (\text{A.23})$$

and

$$\left(\frac{\partial z}{\partial y}\right)_{\psi,x} = \frac{-(\partial\psi/\partial y)_{x,z}}{(\partial\psi/\partial z)_{x,y}} \quad (\text{A.24})$$

Returning to equation A.20 we again put  $dz = 0$ , but we now divide through by  $dy$  rather than by  $dx$

$$0 = \left(\frac{\partial\psi}{\partial x}\right)_{y,z} \left(\frac{\partial x}{\partial y}\right)_{\psi,z} + \left(\frac{\partial\psi}{\partial y}\right)_{x,z} \quad (\text{A.25})$$

whence

$$\left(\frac{\partial x}{\partial y}\right)_{\psi,z} = \frac{-(\partial\psi/\partial y)_{x,z}}{(\partial\psi/\partial x)_{y,z}} \quad (\text{A.26})$$

and, on comparison with equation A.21, we find the very reasonable result that

$$\left(\frac{\partial x}{\partial y}\right)_{\psi,z} = \frac{1}{(\partial y/\partial x)_{\psi,z}} \quad (\text{A.27})$$

From equations A.22 to A.24 we then find

$$\left(\frac{\partial x}{\partial y}\right)_{\psi,z} \left(\frac{\partial y}{\partial z}\right)_{\psi,x} \left(\frac{\partial z}{\partial x}\right)_{\psi,y} = -1 \quad (\text{A.28})$$

Finally we return to our basic equation, which defines the differential  $d\psi$ , and consider the case in which  $x$ ,  $y$ , and  $z$  are themselves functions of a variable  $u$  (as in equation A.12)

$$d\psi = \left[ \left(\frac{\partial \psi}{\partial x}\right)_{y,z} \frac{dx}{du} + \left(\frac{\partial \psi}{\partial y}\right)_{x,z} \frac{dy}{du} + \left(\frac{\partial \psi}{\partial z}\right)_{x,y} \frac{dz}{du} \right] du \quad (\text{A.29})$$

If  $\psi$  is to be constant, there must be a relation among  $x$ ,  $y$ , and  $z$ , hence also among  $dx/du$ ,  $dy/du$ , and  $dz/du$ . We find

$$0 = \left(\frac{\partial \psi}{\partial x}\right)_{y,z} \left(\frac{dx}{du}\right)_{\psi} + \left(\frac{\partial \psi}{\partial y}\right)_{x,z} \left(\frac{dy}{du}\right)_{\psi} + \left(\frac{\partial \psi}{\partial z}\right)_{x,y} \left(\frac{dz}{du}\right)_{\psi} \quad (\text{A.30})$$

If we further require that  $z$  shall be a constant independent of  $u$  we find

$$0 = \left(\frac{\partial \psi}{\partial x}\right)_{y,z} \left(\frac{\partial x}{\partial u}\right)_{\psi,z} + \left(\frac{\partial \psi}{\partial y}\right)_{x,z} \left(\frac{\partial y}{\partial u}\right)_{\psi,z} \quad (\text{A.31})$$

or

$$\frac{(\partial y/\partial u)_{\psi,z}}{(\partial x/\partial u)_{\psi,z}} = -\frac{(\partial \psi/\partial x)_{y,z}}{(\partial \psi/\partial y)_{x,z}} \quad (\text{A.32})$$

Comparison with equation A.22 shows that

$$\left(\frac{\partial y}{\partial x}\right)_{\psi,z} = \frac{(\partial y/\partial u)_{\psi,z}}{(\partial x/\partial u)_{\psi,z}} \quad (\text{A.33})$$

Equations A.22, A.27, and A.33 are among the most useful formal manipulations in thermodynamic calculations.

# APPENDIX B

## MAGNETIC SYSTEMS

If matter is acted on by a magnetic field it generally develops a magnetic moment. A description of this magnetic property, and of its interaction with thermal and mechanical properties, requires the adoption of an additional extensive parameter. This additional extensive parameter  $X$  and its corresponding intensive parameter  $P$  are to be chosen so that the *magnetic work*  $dW_{\text{mag}}$  is

$$dW_{\text{mag}} = P dX \quad (\text{B.1})$$

where

$$dU = dQ + dW_M + dW_c + dW_{\text{mag}} \quad (\text{B.2})$$

Here  $dQ$  is the heat  $T dS$ ,  $dW_M$  is the mechanical work (e.g.,  $-P dV$ ), and  $dW_c$  is the chemical work  $\sum \mu_j dN_j$ . We consider a specific situation that clearly indicates the appropriate choice of parameters  $X$  and  $P$ .

Consider a solenoid, or coil, as shown in Fig. B.1. The wire of which the solenoid is wound is assumed to have zero electrical resistance (superconducting). A battery is connected to the solenoid, and the electromotive force (emf) of the battery is adjustable at will. The thermodynamic system is inside the solenoid, and the solenoid is enclosed within an adiabatic wall.

If no changes occur within the system, and if the current  $I$  is constant, the battery need supply no emf because of the perfect conductivity of the wire.

Let the current be  $I$  and let the local magnetization of the thermodynamic system be  $\mathbf{M}(\mathbf{r})$ . The current  $I$  can be altered at will by controlling the battery emf. The magnetization  $\mathbf{M}(\mathbf{r})$  then will change also. We assume that the magnetization at any position  $\mathbf{r}$  is a single-valued function of the current

$$\mathbf{M}(\mathbf{r}) = \mathbf{M}(\mathbf{r}; I) \quad (\text{B.3})$$

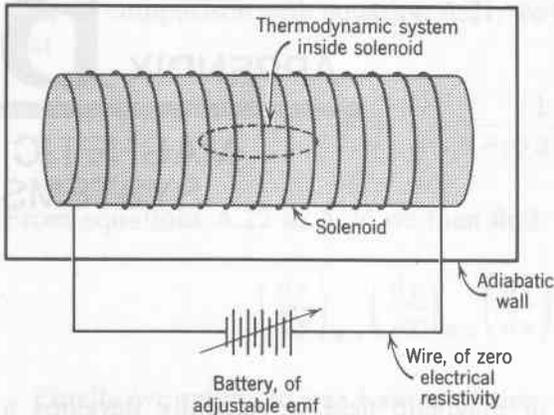


FIGURE B.1

Systems for which  $\mathbf{M}(\mathbf{r}; I)$  is not single valued in  $I$  are said to exhibit *hysteresis*; most ferromagnetic systems have this property. Hysteresis generally is associated with a magnetic heterogeneity of the sample, the separate regions being known as domains. The analysis we shall develop is generally applicable *within* a ferromagnetic domain, but for simplicity we explicitly exclude all hysteretic systems. Paramagnetic, diamagnetic, and antiferromagnetic systems satisfy the requirement that  $\mathbf{M}(\mathbf{r}; I)$  is single valued in  $I$ .

If the thermodynamic system were *not* within the solenoid, the current  $I$  would produce a magnetic field (more accurately, a magnetic *flux density*)  $\mathbf{B}_e(I)$ . This external “field” may be a function of position within the solenoid, but it is linear in  $I$ . That is

$$\mathbf{B}_e = \mathbf{b}I \quad (\text{B.4})$$

where  $\mathbf{b}$  is a vector function of position.

We suppose that the current is increased, thereby increasing the external field  $\mathbf{B}_e$ . The magnetic moment changes in response. In order to accomplish these changes, the battery must deliver work, and we seek the relationship between the work done and the changes in  $\mathbf{B}_e$  and  $\mathbf{M}$ .

The rate at which work is done by the battery is given by

$$\frac{dW_{\text{mag}}}{dt} = I \times (\text{voltage}) \quad (\text{B.5})$$

in which (voltage) denotes the back emf induced in the solenoid windings by the changes that occur within the coil.

The induced emf in the solenoid arises from two sources. One source is independent of the thermodynamic system and results from a change in

the flux associated with the field  $\mathbf{B}_e$ . Rather than compute this flux and voltage, we can write the resultant contribution to  $dW_{\text{mag}}$  directly. For an *empty* solenoid the work is just the change in the energy of the magnetic field, or

$$dW_{\text{mag}} = d\left(\frac{1}{2\mu_0} \int B_e^2 dV\right) \quad (\text{B.6})$$

in which  $\mu_0 = 4\pi \times 10^{-7} \text{T} \cdot \text{m}/\text{A}$ , and in which the integral is taken over the entire volume of the solenoid.

The second contribution to  $dW_{\text{mag}}$  results from the thermodynamic system itself and consequently is of more direct interest to us. It is evident that the change of magnetic moment of each infinitesimal element of the system contributes separately and additively to the total induced emf, and furthermore that the induced emf produced by any change in dipole moment depends not on the nature of the dipole but only on the rate of change of its moment and on its position in the solenoid. Consider then a particular model of an elementary dipole at the position  $\mathbf{r}$ : a small current loop of area  $\mathbf{a}$  and current  $i$ , with a magnetic moment of  $\mathbf{m} = i\mathbf{a}$ . If the current in the solenoid is  $I$ , the field produced by the solenoid at the point  $\mathbf{r}$  is  $\mathbf{B}_e(\mathbf{r}) = \mathbf{b}(\mathbf{r})I$ . This field produces a flux linkage through the small current loop of magnitude  $\mathbf{b}(\mathbf{r}) \cdot \mathbf{a}I$ . Thus the mutual inductance between solenoid and current loop is  $\mathbf{b}(\mathbf{r}) \cdot \mathbf{a}$ . If the current in the current loop changes, it consequently induces a voltage in the solenoid given by

$$(\text{voltage}) = [\mathbf{b}(\mathbf{r}) \cdot \mathbf{a}] \frac{di}{dt} \quad (\text{B.7})$$

$$= \mathbf{b}(\mathbf{r}) \cdot \frac{d\mathbf{m}}{dt} \quad (\text{B.8})$$

$$= \frac{1}{I} \mathbf{B}_e(\mathbf{r}) \cdot \frac{d\mathbf{m}}{dt} \quad (\text{B.9})$$

Thus the work done by the battery is

$$\frac{dW_{\text{mag}}}{dt} = \mathbf{B}_e(\mathbf{r}) \cdot \frac{d\mathbf{m}}{dt} \quad (\text{B.10})$$

Although this result has been obtained for a particular model of an elementary dipole, it holds for any change in elementary dipole moment. In particular if  $\mathbf{M}(\mathbf{r})$  is the *magnetization*, or the dipole moment per unit volume in the system at the point  $\mathbf{r}$ , we set

$$\mathbf{m} = \int \mathbf{M}(\mathbf{r}) dV \quad (\text{B.11})$$

To obtain the total work, we sum over all elementary dipoles, or integrate over the volume of the sample

$$\frac{dW_{\text{mag}}}{dt} = \int \mathbf{B}_e \cdot \frac{d\mathbf{M}}{dt} dV \quad (\text{B.12})$$

Adding the two contributions to the magnetic work, we find

$$dW_{\text{mag}} = d\left(\frac{1}{2\mu_0} \int B_e^2 dV\right) + \int (\mathbf{B}_e \cdot d\mathbf{M}) dV \quad (\text{B.13})$$

This is the fundamental result on which the thermodynamics of magnetic systems is based.

In passing we note that the *local* field  $\mathbf{H}$  can be introduced in place of the *external* field  $\mathbf{H}_e$  by noting that the difference  $\mathbf{H} - \mathbf{H}_e$  is just the field produced by the magnetization  $\mathbf{M}(\mathbf{r})$  acting as a magnetostatic source. In this way it can be shown<sup>1</sup> that

$$dW_{\text{mag}} = \int \mathbf{H} \cdot d\mathbf{B} dV \quad (\text{B.14})$$

where  $\mathbf{H}$  and  $\mathbf{B}$  are *local* values. However the form of the magnetic work expression we shall find most convenient is the first derived (equation B.13).

In the general case the magnetization  $\mathbf{M}(\mathbf{r})$  will vary from point to point within the system, even if the external field  $\mathbf{B}_e$  is constant. This variation may arise from inherent inhomogeneities in the properties of the system, or it may result from demagnetization effects of the boundaries of the system. We wish to develop the theory for homogeneous systems. We therefore assume that  $\mathbf{B}_e$  is constant and that the intrinsic properties of the system are homogeneous. We further assume that the system is ellipsoidal in shape. For such a system the magnetization  $\mathbf{M}$  is independent of position, as shown in any text on magnetostatics.

The magnetic work equation can now be written as

$$dW_{\text{mag}} = d\left(\frac{1}{2\mu_0} \int B_e^2 dV\right) + \mathbf{B}_e \cdot d\mathbf{I} \quad (\text{B.15})$$

where  $\mathbf{I}$  is the total magnetic dipole moment of the system

$$\mathbf{I} = \int \mathbf{M} dV = \mathbf{M}V \quad (\text{B.16})$$

<sup>1</sup>See V. Heine, *Proc. Cambridge Phil. Soc.*, 52, 546 (1956).

The energy differential is

$$d(\text{Energy}) = T dS - P dV + d\left(\frac{1}{2\mu_0} \int B_e^2 dV\right) + \mathbf{B}_e \cdot d\mathbf{I} + \sum_1^r \mu_j dN_j \quad (\text{B.17})$$

The third term on the right of the foregoing equation does not involve the thermodynamic system itself but arises only from the magnetostatic energy of the empty solenoid. Consequently it is convenient to absorb this term into the definition of the energy. We define the energy  $U$  by

$$U \equiv \text{Energy} - \frac{1}{2\mu_0} \int B_e^2 dV \quad (\text{B.18})$$

so that  $U$  is the total energy contained within the solenoid relative to the state in which the system is removed to its field free fiducial state and the solenoid is left with the field  $\mathbf{B}_e$ . This redefinition of the internal energy does not alter any of the formalism of thermodynamics. Thus we write

$$dU = T dS - P dV + B_e dI_B + \sum_1^r \mu_j dN_j \quad (\text{B.19})$$

where  $I_B$  is the component of  $\mathbf{I}$  parallel to  $\mathbf{B}_e$ .

*The extensive parameter descriptive of the magnetic properties of a system is  $\mathbf{I}_B$ , the component of the total magnetic moment parallel to the external field. The intensive parameter in the energy representation is  $\mathbf{B}_e$ .*

The fundamental equation is

$$U = U(S, V, I_B, N_1, \dots, N_r) \quad (\text{B.20})$$

and

$$\left(\frac{\partial U}{\partial I_B}\right)_{S, V, N_1, \dots, N_r} = B_e \quad (\text{B.21})$$

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### CONCEPTUAL OVERVIEWS

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## UNITS AND CONVERSION FACTORS

### Energy

1 Joule =	
10 <sup>7</sup> ergs	(10 <sup>-7</sup> J/erg)
0.2389 calories	(4.186 J/cal)
9.480 × 10 <sup>-4</sup> Btu	(1055 J/Btu)
9.869 × 10 <sup>-3</sup> liter-atmospheres	(101.3 J/liter-atm)
0.7376 foot-pounds	(1.356 J/ft-pound)
2.778 × 10 <sup>-4</sup> watt-hours	(3600 J/watt-hr)
3.724 × 10 <sup>-7</sup> horsepower-hours	(2.685 × 10 <sup>6</sup> J/hp-hr)

### Pressure

1 Pascal =	
1 Newton/m <sup>2</sup>	
10 dynes/cm <sup>2</sup>	
10 baryes	
10 <sup>-5</sup> bars	
1.450 × 10 <sup>-4</sup> psi (pounds/inch <sup>2</sup> )	(6897 Pa/psi)
0.9869 × 10 <sup>-5</sup> atmospheres	(1.013 × 10 <sup>5</sup> Pa/atm)
7.5006 × 10 <sup>-3</sup> Torr (or "mm Hg")	(133.3 Pa/Torr)

### Volume

1 m <sup>3</sup> =	
10 <sup>6</sup> cm <sup>3</sup>	
10 <sup>3</sup> liters	
6.1024 × 10 <sup>4</sup> inch <sup>3</sup>	(1.639 × 10 <sup>-5</sup> m <sup>3</sup> /in <sup>3</sup> )
35.315 ft <sup>3</sup>	(.02832 m <sup>3</sup> /ft <sup>3</sup> )
264.17 U.S. gallons	(3.785 × 10 <sup>-3</sup> m <sup>3</sup> /gal)
219.97 British Imperial gallons	(4.546 × 10 <sup>-3</sup> m <sup>3</sup> /gal)

### Temperature

$$T(^{\circ}\text{C}) = T(\text{Kelvin}) - 273.15$$

$$T(^{\circ}\text{R}) = 1.8 \times T(\text{Kelvin})$$

$$T(^{\circ}\text{F}) = T(^{\circ}\text{R}) - 459.67 = 1.8 \times T(^{\circ}\text{C}) + 32$$

---

## CONSTANTS

$$R = 8.314 \text{ Joule/mole-Kelvin} = 1.986 \text{ calories/mole-Kelvin}$$

$$k_B = 1.381 \times 10^{-23} \text{ Joules/Kelvin}$$

$$N_A = R/k_B = 6.022 \times 10^{23} / \text{mole}$$

$$h = 6.626 \times 10^{-34} \text{ Joule-sec}$$